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Vertex-algebraic structure of the principal subspaces of level one modules for the untwisted affine Lie algebras of types A , D , E

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ABSTRACT

Generalizing some of our earlier work, we prove natural presentations of the principal subspaces of the level one standard modules for the untwisted affine Lie algebras of types A , D and E , and also of certain related spaces. As a consequence, we obtain a canonical complete set of recursions (q -difference equations) for the (multi-)graded dimensions of these spaces, and we derive their graded dimensions. Our methods are based on intertwining operators in vertex operator algebra theory.

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1. Introduction

This paper is a continuation of [CalLM1] and [CalLM2]. Our aim is to continue the study of the structures called principal subspaces, which had been introduced by Feigin and Stoyanovsky [FS1], [FS2], and related spaces, but now in the case of the basic (= level-one standard) modules for the affine Lie algebras of types A , D and E . Specifically, we prove a generators-and-relations result (a presentation) for these principal subspaces, and exploit this result to construct exact sequences and recursion relations yielding their (multi-)graded dimensions (= generating functions of the dimensions of the homogeneous subspaces, sometimes called “characters”). Our methods are based on intertwining operators in vertex operator algebra theory.

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Compared to the $\widehat{\mathfrak{sl}(2)}$ case that was handled in [CalLM1] and [CalLM2] (as well as in [CLM1] and [CLM2]), the vertex-algebraic structure associated with the principal subspaces in the present generality is more complex than the corresponding structure in these earlier papers, and this makes the proofs more subtle. For a detailed introduction to and motivation of these ideas, including historical references, we refer the reader to [CLM1], [CLM2], [Cal1], [Cal2], [CalLM1] and [CalLM2], and Feigin and Stoyanovsky's papers [FS1], [FS2]. Presentations of principal subspaces have also been considered in [AKS], [AK] and [FF], and principal subspaces and related structures have also been studied in [P1], [G], [P2], [FF]MM and other works.

In particular, our own interest in principal subspaces arose from the idea developed and implemented in [CLM1], [CLM2] that one could “explain” the classical Rogers–Ramanujan and Rogers–Selberg recursions for the “sum sides” of the Rogers–Ramanujan and Gordon–Andrews partition identities (cf. [A]) by means of exact sequences and recursions, constructed from intertwining operators in vertex operator algebra theory, associated with the principal subspaces of the standard $\widehat{\mathfrak{sl}(2)}$ -modules.

Let us recall the notion of principal subspace. Let \mathfrak{g} be a finite-dimensional (complex) simple Lie algebra of type A , D or E . Fix a dominant weight Λ for $\hat{\mathfrak{g}}$ and consider the standard (integrable highest weight) $\hat{\mathfrak{g}}$ -module $L(\Lambda)$. Then the principal subspace $W(\Lambda) \subset L(\Lambda)$ is defined as

$$W(\Lambda) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda},$$

where v_{Λ} is a highest weight vector of $L(\Lambda)$, $\mathfrak{n} \subset \mathfrak{g}$ is the Lie subalgebra of \mathfrak{g} spanned by the root vectors for the positive roots, and $\bar{\mathfrak{n}}$ is the appropriate affinization of \mathfrak{n} in $\hat{\mathfrak{g}}$. Then

$$W(\Lambda) \simeq U(\bar{\mathfrak{n}}) / \text{Ker } f_{\Lambda},$$

where

$$f_{\Lambda} : U(\bar{\mathfrak{n}}) \longrightarrow W(\Lambda)$$

is the natural surjection that takes an element a to $a \cdot v_{\Lambda}$.

Certain well-understood formal infinite sums of elements of $\bar{\mathfrak{n}}$ are well known to annihilate the standard module $L(\Lambda)$, and thus natural truncations of these formal infinite sums lie in $\text{Ker } f_{\Lambda}$. The nontrivial part of proving the desired presentation (the hard part) is to prove that these truncated sums, together with obvious additional elements, *generate* $\text{Ker } f_{\Lambda}$. (As is typically the case with generators-and-relations results in mathematics, the hard part is to prove that the “known” relations generate *all* of the relations defining the structure being studied.)

For the case $\mathfrak{g} = \mathfrak{sl}(2)$, the appropriate presentations of the principal subspaces of the standard $\hat{\mathfrak{g}}$ -modules of all levels were given in [FS1], [FS2] (and were invoked in the course of the proofs of the main theorems in [CLM1], [CLM2]). If one already has available an appropriate, explicit “fermionic character formula” for a standard module, then it is relatively straightforward to show that the formal infinite sums referred to above generate *all* the relations defining the principal subspace, and indeed, such an explicit fermionic character formula (as it came to be called) had been discovered in [LP2], providing enough information to justify the desired presentation of the principal subspaces for the standard $\widehat{\mathfrak{sl}(2)}$ -modules. The fermionic character formula in [LP2] was in fact proved by means of the construction of what came to be called a “fermionic basis” for each standard $\widehat{\mathfrak{sl}(2)}$ -module. These bases, which were untwisted analogues of the fermionic bases constructed in [LW1], [LW2], [LW3], [LW4] (and used in those works to give vertex-operator proofs of the Rogers–Ramanujan identities and vertex-operator interpretations of the Gordon–Andrews–Bressoud identities), were motivated by the discovery and use of the formal infinite sums mentioned above; these formal infinite sums were used to construct natural spanning sets of the standard $\widehat{\mathfrak{sl}(2)}$ -modules (see also [LP1]). The harder part of the work required in [LP2] to complete the proof of the main theorem constructing the fermionic bases was to prove the *linear independence* of the spanning sets that had been constructed, and as we have mentioned, the construction of these fermionic bases (including of course the proof of their

linear independence) and the resulting fermionic character formulas allow one to prove the nontrivial part of the expected presentation result for the principal subspaces quite easily. Beyond the case of $\mathfrak{sl}(2)$, it is typically difficult to construct fermionic bases in general.

The main point of our work beginning in [CalLM1] and [CalLM2], and continuing in the present paper, is to prove the nontrivial part of the presentation of the principal subspace *without* invoking a theorem (or perhaps conjecture) such as a known (or perhaps proposed) construction of a fermionic basis or a known (or proposed) fermionic character formula. Rather, what one really wants to do is to provide a priori proofs of the desired presentations of the principal subspaces, and then to use the presentation result in the course of the construction of exact sequences and recursions whose solutions will yield, as theorems, fermionic character formulas and fermionic bases. When one combines the results of [CalLM1] and [CalLM2] with those of [CLM1] and [CLM2], one indeed has an a priori derivation of the desired fermionic character formulas, without the use of explicit fermionic bases or fermionic character formulas such as those derived in [LP2].

Specifically, the main purpose of the present paper is to give an a priori proof of the expected presentations of the principal subspaces of the level one standard $\hat{\mathfrak{g}}$ -modules for types A , D or E . Our proof is a generalization of our previous a priori proof of this presentation for the case $\mathfrak{g} = \mathfrak{sl}(2)$ carried out in [CalLM1], and our methods continue the development of the vertex-operator-algebraic ideas of [CalLM1] and [CalLM2]. Our arguments are quite delicate, and this seems to be necessary.

When \mathfrak{g} is of type A , D or E and the standard module $L(\Lambda)$ is of level one, the subspace $W(\Lambda)$ can be realized inside the module V_P for the lattice vertex operator algebra V_Q , where Q is the root lattice and P is the weight lattice of \mathfrak{g} , and it is convenient for us to use this well-known realization in this paper.

The main results of this paper are, for \mathfrak{g} of type A , D or E :

- (a) Proof of the expected presentations of the principal subspaces for all the basic modules; that is, we explicitly describe the left ideal $\text{Ker } f_\Lambda$, showing that its “obvious” elements indeed generate it (Theorem 4.2).
- (b) Construction of certain canonical exact sequences among the principal subspaces considered in (a) (Theorem 5.1).
- (c) Explicit formulas for the graded dimensions of the spaces $W(\Lambda)$ (Corollary 5.1).
- (d) A reformulation of (a) in terms of two-sided ideals of a suitable completion of $U(\bar{\mathfrak{n}})$ (Theorem 4.3).

In fact, we in addition obtain analogues of these results for certain spaces that we call “principal-like” subspaces of the V_Q -module V_P ; these subspaces arose naturally, and in fact were required, in the course of our proofs of these results (for principal subspaces). It is interesting that these principal-like subspaces are closely related to versions of Kirillov–Reshetikhin modules, whose relevance to the study of principal subspaces and so on was discussed in [AK].

Explicit formulas such as those in (c) had already been proposed and studied in the literature, in particular, in [DKKMM], [KKMM1], [KKMM2], [KNS] and [T], in the setting of the thermodynamic Bethe Ansatz, and formulas of this type had been linked to principal subspaces in [FS1], [FS2], and further, in [G], [AKS], [Cal1] and [FFJMM]. Motivated by [FS1], such formulas for the principal subspaces of certain classes of standard modules for type A were proved in [G], and were proved by a different method more recently in [Cal1] for type A , level 1, a method that also included (a) and (b). But the main theorems in the present paper had not been proved before in the present generality.

In the spirit of our earlier papers [CalLM1], [CalLM2] (and also in the spirit of [CLM1], [CLM2]), we obtain our results referred to in (a)–(d) in an a priori way, by combining vertex-algebraic methods with general facts about affine Lie algebras, and without any reference to spanning sets or bases of $W(\Lambda)$, or to any fermionic formulas for graded dimensions.

In G. Georgiev’s paper [G], which appeared shortly after [FS1], his goal was essentially combinatorial, even though vertex operator techniques were extensively used; combinatorial bases of principal subspaces were obtained in the case of special families of standard modules of type A . Not surprisingly, our formulas in (c), for type A , coincide with Georgiev’s formulas, although our methods are

quite different. (Georgiev was not concerned with formulating or proving results analogous to those in (a), (b) or (d).)

The paper by E. Ardonne, R. Kedem and M. Stone [AKS] gives, among other things, a general formula for the graded dimension of $W(\Lambda)$ for all standard modules for $\widehat{\mathfrak{sl}(n+1)}$. The method used to justify this formula in [AKS] is reminiscent of the method that Feigin and Stoyanovsky pursued in [FS1], where a suitable dual of $W(\Lambda)$ was described in terms of certain rational functions (see Theorem 3.4 of [AKS]); this description can be used to give a combinatorial interpretation of graded dimensions. The authors of [AKS] and [AK] quote a known, natural presentation of a standard module, based on the formal infinite sums that we have been referring to, and argue that the subset of this set of formal infinite sums that relate in a direct way to the principal subspace must give a complete set of defining relations for the principal subspace (but it is a priori possible that the other relations needed for the presentation of the standard module could in principle contribute to further relations needed for a correct presentation of the principal subspace, and it seems to us that the necessity to exclude this possibility requires that a proof of the presentation of $W(\Lambda)$ be nontrivial). In [FFJMM], B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin prove the presentation of $W(\Lambda)$ for the case of $\widehat{\mathfrak{sl}(3)}$, all levels, in the course of establishing a formula for the graded dimension. In the paper [FF], B. Feigin and E. Feigin discuss lattice vertex operator algebras, and when the Gram matrix has only nonnegative integer entries (cf. the text before Lemma 1.1 in [FF]), a presentation result for spaces analogous to principal subspaces is asserted, and a comparison of graded dimensions is invoked to justify this. Principal subspaces for general lattice vertex operator algebras and superalgebras, and their modules and twisted modules, are being studied in [MPe].

Our paper, then, fills what appear to be some gaps in the literature concerning principal subspaces, but what is more interesting to us is that our methods are natural (if perhaps also rather subtle), and they generalize considerably, as ongoing work seems to be showing.

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2. Preliminaries

Let \mathfrak{g} be a finite-dimensional complex simple Lie algebra of type A , D or E of rank l , and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $\{\alpha_1, \dots, \alpha_l\} \subset \mathfrak{h}^*$ be a set of simple roots of \mathfrak{g} . Denote by Δ the set of roots and by Δ_+ the set of positive roots. We use the rescaled Killing form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} such that $\langle \alpha, \alpha \rangle = 2$ for $\alpha \in \Delta$, where we identify \mathfrak{h} with \mathfrak{h}^* via this form. For each root α fix a root vector x_α , to be rescaled later.

Denote by $\lambda_1, \dots, \lambda_l \in \mathfrak{h} \simeq \mathfrak{h}^*$ the corresponding fundamental weights of \mathfrak{g} (i.e., $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $i, j = 1, \dots, l$). It is also convenient to set

$$\lambda_0 = 0.$$

Let $Q = \sum_{i=1}^l \mathbb{Z}\alpha_i \subset \mathfrak{h} \simeq \mathfrak{h}^*$ and $P = \sum_{i=1}^l \mathbb{Z}\lambda_i \subset \mathfrak{h} \simeq \mathfrak{h}^*$ be the root and weight lattices of \mathfrak{g} , respectively. If \mathfrak{g} of type A_l , $l \geq 1$, we have $P/Q \simeq \mathbb{Z}/(l+1)\mathbb{Z}$. For \mathfrak{g} of type D_l , $l \geq 4$, we have $P/Q \simeq \mathbb{Z}/4\mathbb{Z}$ or $P/Q \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, depending on whether l is odd or even. If \mathfrak{g} is E_6 , E_7 or E_8 the group P/Q is $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or the trivial group, respectively.

Consider the untwisted affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}, \quad (2.1)$$

where \mathbf{k} is a nonzero central element and

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\langle x, y \rangle \delta_{m+n, 0} \mathbf{k}$$

for $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. By adjoining the degree operator d ($[d, x \otimes t^m] = m$, $[d, \mathbf{k}] = 0$) to the Lie algebra $\hat{\mathfrak{g}}$ one obtains the affine Kac–Moody algebra $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \oplus \mathbb{C}d$ (cf. [K]).

Set

$$\mathfrak{n} = \coprod_{\alpha \in \Delta_+} \mathbb{C}x_\alpha \quad (2.2)$$

and consider the subalgebras

$$\begin{aligned} \bar{\mathfrak{n}} &= \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}], \\ \bar{\mathfrak{n}}_+ &= \mathfrak{n} \otimes \mathbb{C}[t] \end{aligned}$$

and

$$\bar{\mathfrak{n}}_- = \mathfrak{n} \otimes t^{-1}\mathbb{C}[t^{-1}]$$

of $\hat{\mathfrak{g}}$. We shall frequently use the decomposition

$$U(\bar{\mathfrak{n}}) = U(\bar{\mathfrak{n}}_-) \oplus U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+. \quad (2.3)$$

The affine Lie algebra $\hat{\mathfrak{g}}$ has the decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_{<0} \oplus \hat{\mathfrak{g}}_{\geq 0},$$

where

$$\hat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$$

and

$$\hat{\mathfrak{g}}_{\geq 0} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k}.$$

The form $\langle \cdot, \cdot \rangle$ on \mathfrak{g} extends naturally to $\mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d$, with $\langle \mathbf{k}, d \rangle = 1$. We shall identify $\mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d$ with its dual space $(\mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d)^*$ via this form. As usual, we denote by $\alpha_0, \alpha_1, \dots, \alpha_l \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d)^*$ the corresponding simple roots, and by $\Lambda_0, \Lambda_1, \dots, \Lambda_l \in (\mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d)^*$ the corresponding fundamental weights. Then $\langle \Lambda_0, \mathbf{k} \rangle = 1$; for $i = 1, \dots, l$, $\langle \Lambda_i, \mathbf{k} \rangle = k_i$, where $k_i \geq 1$ is the coefficient of α_i in the expansion of the highest root; $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ for $i, j = 0, \dots, l$; and

$$\langle \Lambda_0, d \rangle = 0, \quad \langle \Lambda_i, d \rangle = -\frac{1}{2} \langle \Lambda_i, \Lambda_i \rangle \quad \text{for } i = 1, \dots, l. \quad (2.4)$$

Moreover, for $i = 0, \dots, l$,

$$\Lambda_i = \lambda_i + \langle \Lambda_i, d \rangle \mathbf{k} + \langle \Lambda_i, \mathbf{k} \rangle d,$$

so that in particular, $\Lambda_0 = d$. In this paper we shall focus on the level one standard $\hat{\mathfrak{g}}$ -modules (also called the basic $\hat{\mathfrak{g}}$ -modules), denoted $L(\Lambda)$, where Λ is one of the fundamental weights Λ_i such that $\langle \Lambda, \mathbf{k} \rangle = 1$. Let v_Λ be a highest weight vector of such a module $L(\Lambda)$. We shall normalize v_Λ later. We view each of the basic $\hat{\mathfrak{g}}$ -modules as a $\tilde{\mathfrak{g}}$ -module, where d acts according to (2.4) on v_Λ . The order of P/Q gives the number of inequivalent standard $\hat{\mathfrak{g}}$ -modules of level one: For $\mathfrak{g} = A_l$ there are $l+1$ level one standard $\hat{\mathfrak{g}}$ -modules: $L(\Lambda_0), L(\Lambda_1), \dots, L(\Lambda_l)$. For $\mathfrak{g} = D_l$ there are four such modules: $L(\Lambda_0), L(\Lambda_1), L(\Lambda_{l-1})$ and $L(\Lambda_l)$. If \mathfrak{g} is E_6 there are three: $L(\Lambda_0), L(\Lambda_1)$ and $L(\Lambda_6)$. There are two basic $\hat{\mathfrak{g}}$ -modules, $L(\Lambda_0)$ and $L(\Lambda_1)$, when \mathfrak{g} is E_7 , and only one such module, $L(\Lambda_0)$, if \mathfrak{g} is E_8 . (Cf. [K].)

For each fundamental weight Λ_i of $\hat{\mathfrak{g}}$ such that $\langle \Lambda_i, \mathbf{k} \rangle = 1$ consider the principal subspace $W(\Lambda_i)$ of the level one standard module $L(\Lambda_i)$ in the sense of [FS1], [FS2] (see also [CalLM2] for a natural generalization of this notion):

$$W(\Lambda_i) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda_i}. \quad (2.5)$$

By the highest weight vector property we have

$$W(\Lambda_i) = U(\bar{\mathfrak{n}}_-) \cdot v_{\Lambda_i}. \quad (2.6)$$

As in [CalLM1], [CalLM2] we consider the surjective maps

$$\begin{aligned} F_{\Lambda_i} : U(\hat{\mathfrak{g}}) &\longrightarrow L(\Lambda_i), \\ a &\longmapsto a \cdot v_{\Lambda_i}. \end{aligned} \quad (2.7)$$

Restrict F_{Λ_i} to $U(\bar{\mathfrak{n}})$ and denote these (surjective) restrictions by f_{Λ_i} :

$$\begin{aligned} f_{\Lambda_i} : U(\bar{\mathfrak{n}}) &\longrightarrow W(\Lambda_i), \\ a &\longmapsto a \cdot v_{\Lambda_i}. \end{aligned} \quad (2.8)$$

In this paper we will give a precise description of the kernels $\text{Ker } f_{\Lambda_i}$, and thus a presentation of the principal subspaces $W(\Lambda_i)$.

Remark 2.1. In [CalLM1] and [CalLM2], we used the symbol f_{Λ} for the restriction of F_{Λ} to $U(\bar{\mathfrak{n}}_-)$ rather than to $U(\bar{\mathfrak{n}})$, but since $\bar{\mathfrak{n}}$ is no longer abelian in general, the present maps f_{Λ} are the appropriate ones.

As in [CalLM1], [CalLM2] we will sometimes use generalized Verma modules for $\hat{\mathfrak{g}}$, in the sense of [L1], [GL], [L2], in our formulations and proofs. The generalized Verma module $N(\Lambda_0)$ is defined as the induced $\hat{\mathfrak{g}}$ -module

$$N(\Lambda_0) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} \mathbb{C} v_{\Lambda_0}^N, \quad (2.9)$$

where $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially and \mathbf{k} acts as the scalar 1 on $\mathbb{C} v_{\Lambda_0}^N$; $v_{\Lambda_0}^N$ is a highest weight vector. More generally, for any fundamental weight Λ_i such that $\langle \Lambda_i, \mathbf{k} \rangle = 1$, define the generalized Verma module

$$N(\Lambda_i) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} U_i \simeq U(\hat{\mathfrak{g}}_{< 0}) \otimes U_i, \quad (2.10)$$

where U_i is a copy of the finite-dimensional irreducible \mathfrak{g} -module $U(\mathfrak{g}) \cdot v_{\Lambda_i} \subset L(\Lambda_i)$, with highest weight vector now called $v_{\Lambda_i}^N$, and where $\mathfrak{g} \otimes t\mathbb{C}[t]$ acts trivially and \mathbf{k} by 1. We view all these generalized Verma modules as $\tilde{\mathfrak{g}}$ -modules, where d acts according to (2.4) on $v_{\Lambda_i}^N$. Continuing to generalize [CalLM1], [CalLM2], we introduce

$$W^N(\Lambda_i) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda_i}^N \simeq U(\bar{\mathfrak{n}}_-) \cdot v_{\Lambda_i}^N, \quad (2.11)$$

the principal subspace of $N(\Lambda_i)$; this is naturally an $\bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d$ -submodule of $N(\Lambda_i)$. We have the natural surjective $\hat{\mathfrak{g}}$ -module maps

$$\begin{aligned} F_{\Lambda_i}^N : U(\hat{\mathfrak{g}}) &\longrightarrow N(\Lambda_i), \\ a &\longmapsto a \cdot v_{\Lambda_i}^N \end{aligned} \quad (2.12)$$

and their restrictions to $U(\bar{\mathfrak{n}})$,

$$\begin{aligned} f_{\Lambda_i}^N : U(\bar{\mathfrak{n}}) &\longrightarrow W^N(\Lambda_i), \\ a &\longmapsto a \cdot v_{\Lambda_i}^N, \end{aligned} \quad (2.13)$$

where $\langle \Lambda_i, \mathbf{k} \rangle = 1$. For such fundamental weights Λ_i we have the natural surjective $\tilde{\mathfrak{g}}$ -module maps

$$\begin{aligned} \Pi_{\Lambda_i} : N(\Lambda_i) &\longrightarrow L(\Lambda_i), \\ a \cdot v_{\Lambda_i}^N &\longmapsto a \cdot v_{\Lambda_i} \end{aligned} \quad (2.14)$$

for $a \in U(\hat{\mathfrak{g}})$ and their restrictions to $W^N(\Lambda_i)$,

$$\pi_{\Lambda_i} : W^N(\Lambda_i) \longrightarrow W(\Lambda_i); \quad (2.15)$$

these are $U(\bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d)$ -module surjections.

Throughout this paper we will write $x(m)$ for the action of $x \otimes t^m \in \hat{\mathfrak{g}}$ on any $\hat{\mathfrak{g}}$ -module, for $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$. In particular, we have the operator $x_\alpha(m)$, the image of $x_\alpha \otimes t^m$, for $\alpha \in \Delta$. Sometimes we will write $x(m)$ for the Lie algebra element $x \otimes t^m$ itself; it will be clear from the context whether $x(m)$ is an operator or a Lie algebra element.

Now we recall the constructions of lattice vertex operator algebras and their modules from Section 7.1 and Chapter 8 of [FLM2] (cf. [FLM1], [B] and Sections 6.4 and 6.5 of [LL]). We work in the setting of [LL], adapted to our situation. Consider $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ and its irreducible induced module

$$M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}\mathbf{k})} \mathbb{C}, \quad (2.16)$$

where $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially and \mathbf{k} acts as 1 on the one-dimensional module \mathbb{C} . The space $M(1)$ can be identified with the symmetric algebra $S(\hat{\mathfrak{h}}_-)$, where

$$\hat{\mathfrak{h}}_- = \mathfrak{h} \otimes t^{-1} \mathbb{C}[t^{-1}].$$

We shall fix $s > 0$ and a central extension \hat{P} of the weight lattice P (and by restriction this gives a central extension of the root lattice Q) by the finite cyclic group $\langle \kappa \rangle = \langle \kappa \mid \kappa^s = 1 \rangle$ of order s ,

$$1 \longrightarrow \langle \kappa \rangle \longrightarrow \hat{P} \xrightarrow{\sim} P \longrightarrow 1,$$

satisfying the condition (2.17) below. Let $c_0 : P \times P \longrightarrow \mathbb{Z}/s\mathbb{Z}$ be the associated commutator map, so that $aba^{-1}b^{-1} = \kappa^{c_0(\bar{a}, \bar{b})}$ for $a, b \in \hat{P}$, let $v_s \in \mathbb{C}^\times$ be a primitive s th root of unity, and define the map $c : P \times P \longrightarrow \mathbb{C}^\times$ by $c(\alpha, \beta) = v_s^{c_0(\alpha, \beta)}$ for $\alpha, \beta \in P$. Then $c(\cdot, \cdot)$ is an alternating \mathbb{Z} -bilinear map from $P \times P$ to the multiplicative group \mathbb{C}^\times . We assume the condition

$$c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle} \quad \text{for } \alpha, \beta \in Q; \quad (2.17)$$

there indeed exists $s > 0$ together with such a central extension \hat{P} (see Remark 6.4.12 in [LL]).

Define the faithful character $\chi : \langle \kappa \rangle \longrightarrow \mathbb{C}^\times$ by the condition $\chi(\kappa) = v_s$, and denote by $\mathbb{C}\{P\}$ the induced \hat{P} -module $\mathbb{C}[\hat{P}] \otimes_{\mathbb{C}[\langle \kappa \rangle]} \mathbb{C}_\chi$, where \mathbb{C}_χ is the one-dimensional space \mathbb{C} viewed as a $\langle \kappa \rangle$ -module (i.e., $\kappa \cdot 1 = v_s$). Then the space

$$V_Q = M(1) \otimes \mathbb{C}\{Q\}$$

carries a natural vertex operator algebra structure, with 1 as vacuum vector, and the space

$$V_P = M(1) \otimes \mathbb{C}\{P\}$$

is a V_Q -module in a natural way, as specified in [FLM2] and Sections 6.4 and 6.5 of [LL], in particular, Theorems 6.5.1, 6.5.3 and 6.5.20 of [LL]. For the “well known,” but nontrivial, natural uniqueness of these structures of vertex operator algebra and module, see Remark 6.5.4, Proposition 6.5.5, Remark 6.5.6 and Remark 6.5.25 of [LL].

We recall certain features of this structure from [LL]. For convenience, choose a section

$$\begin{aligned} e : P &\longrightarrow \widehat{P}, \\ \alpha &\longmapsto e_\alpha, \end{aligned} \tag{2.18}$$

normalized by the condition $e_0 = 1$, and denote by $\epsilon_0 : P \times P \longrightarrow \mathbb{Z}/s\mathbb{Z}$ the corresponding 2-co-cycle, defined by the condition $e_\alpha e_\beta = \kappa^{\epsilon_0(\alpha, \beta)} e_{\alpha+\beta}$ for $\alpha, \beta \in P$. Define $\epsilon : P \times P \longrightarrow \mathbb{C}^\times$ by $\epsilon(\alpha, \beta) = \nu_s^{\epsilon_0(\alpha, \beta)}$. Then for any $\alpha, \beta \in P$ we have

$$\epsilon(\alpha, \beta)/\epsilon(\beta, \alpha) = c(\alpha, \beta) \tag{2.19}$$

and

$$\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1. \tag{2.20}$$

The choice of the section (2.18) allows us to identify $\mathbb{C}\{P\}$ with the group algebra $\mathbb{C}[P]$, viewed as a vector space, by the linear isomorphism

$$\begin{aligned} \mathbb{C}[P] &\longrightarrow \mathbb{C}\{P\}, \\ e^\alpha &\longmapsto \iota(e_\alpha) \end{aligned} \tag{2.21}$$

for $\alpha \in P$, where, for $a \in \widehat{P}$, we set $\iota(a) = a \otimes 1 \in \mathbb{C}\{P\}$. The action of \widehat{P} on $\mathbb{C}[P]$ is given by $e_\alpha \cdot e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}$, $\kappa \cdot e^\beta = \nu_s e^\beta$ for $\alpha, \beta \in P$, and as operators on $\mathbb{C}[P] \simeq \mathbb{C}\{P\}$ we have

$$e_\alpha e_\beta = \epsilon(\alpha, \beta) e_{\alpha+\beta}. \tag{2.22}$$

We also have the identification $\mathbb{C}[Q] \simeq \mathbb{C}\{Q\}$ and the identifications

$$V_Q = M(1) \otimes \mathbb{C}[Q]$$

and

$$V_P = M(1) \otimes \mathbb{C}[P].$$

For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$, we have the standard operators $h(n)$ on V_P (recall formulas (6.4.47) and (6.4.48) in [LL]), providing V_P with $\widehat{\mathfrak{h}}$ -module structure. The module V_P for the vertex operator algebra V_Q is the direct sum of the irreducible V_Q -modules $M(1) \otimes \mathbb{C}[Q] e^{\lambda_i}$, where i ranges through the indices such that $\langle A_i, \mathbf{k} \rangle = 1$. For all $i = 0, \dots, l$, including those for which $\langle A_i, \mathbf{k} \rangle = 1$, set

$$V_Q e^{\lambda_i} = M(1) \otimes \mathbb{C}[Q] e^{\lambda_i}. \tag{2.23}$$

For $\lambda \in P$ we have the vertex operator

$$Y(\iota(e_\lambda), x) = E^-(-\lambda, x)E^+(-\lambda, x)e_\lambda x^\lambda \quad (2.24)$$

(see formula (6.4.65) in [LL]), where

$$E^\pm(-\lambda, x) = \exp\left(\sum_{\pm n > 0} \frac{-\lambda(n)}{n} x^{-n}\right) \in (\text{End } V_P)[[x, x^{-1}]]$$

and the operator x^λ is defined by

$$x^\lambda(v \otimes \iota(e_\beta)) = x^{(\lambda, \beta)}(v \otimes \iota(e_\beta))$$

for $v \in M(1)$ and $\beta \in P$. Using the identification (2.21) we shall write $Y(e^\lambda, x)$ instead of $Y(\iota(e_\lambda), x)$, for convenience. In particular, for any $\alpha \in \Delta$ we have the operators $x_\alpha(m)$ defined by

$$Y(e^\alpha, x) = \sum_{m \in \mathbb{Z}} x_\alpha(m) x^{-m-1}. \quad (2.25)$$

These operators together with the action of $\hat{\mathfrak{h}}$ give V_P a $\hat{\mathfrak{g}}$ -module structure, and we identify $x_\alpha(0)$ with the root vector $x_\alpha \in \mathfrak{g}$. Recall from Proposition 6.4.5 of [LL] that

$$x^\lambda e_\beta = x^{(\lambda, \beta)} e_\beta x^\lambda \quad (2.26)$$

and

$$\lambda(m) e_\beta = e_\beta \lambda(m) \quad (2.27)$$

for all $\lambda, \beta \in P$ and $m \in \mathbb{Z}$. Using (2.19), (2.22) and (2.24)–(2.27) we obtain, for $\alpha \in \Delta$,

$$x_\alpha(m) e_\beta = c(\alpha, \beta) e_\beta x_\alpha(m + \langle \alpha, \beta \rangle). \quad (2.28)$$

We take

$$\omega = \frac{1}{2} \sum_{i=1}^l u^{(i)} (-1)^2 \mathbf{1}$$

for the standard conformal vector, where $\{u^{(i)}, \dots, u^{(l)}\}$ is an orthonormal basis of \mathfrak{h} (recall formula (6.4.9) in [LL]), so that the operators $L(n)$ defined by

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n) x^{-n-2} \quad (2.29)$$

provide a representation of the Virasoro algebra of central charge l .

For each fundamental weight Λ_i with $\langle \Lambda_i, \mathbf{k} \rangle = 1$ we may and do identify the level one standard $\hat{\mathfrak{g}}$ -module $L(\Lambda_i)$ with the irreducible V_Q -submodule $V_Q e^{\Lambda_i}$ of V_P (recall (2.23)), so that in particular $L(\Lambda_0) = V_Q$, and we take as its highest weight vector

$$v_{\Lambda_i} = e^{\Lambda_i}; \quad (2.30)$$

in particular,

$$v_{\Lambda_0} = 1.$$

For $\mathfrak{g} = A_l$, the irreducible $L(\Lambda_0)$ -modules (up to isomorphism) are $L(\Lambda_0), \dots, L(\Lambda_l)$. For $\mathfrak{g} = D_l$ the spaces $L(\Lambda_0), L(\Lambda_1), L(\Lambda_{l-1})$ and $L(\Lambda_l)$ are the irreducible $L(\Lambda_0)$ -modules. For $\mathfrak{g} = E_6$, $L(\Lambda_0), L(\Lambda_1)$ and $L(\Lambda_6)$ are the irreducible $L(\Lambda_0)$ -modules. When $\mathfrak{g} = E_7$, $L(\Lambda_0)$ and $L(\Lambda_1)$ are the irreducible $L(\Lambda_0)$ -modules, and for $\mathfrak{g} = E_8$, $L(\Lambda_0)$ is the only irreducible $L(\Lambda_0)$ -module. (See [D], [DL], [DLM], [LL].)

The generalized Verma module $N(\Lambda_0)$ carries a natural structure of vertex operator algebra with $v_{\Lambda_0}^N$ as vacuum vector and with central charge l (see Theorem 6.2.18 in [LL]), and for each fundamental weight Λ_i such that $\langle \Lambda_i, \mathbf{k} \rangle = 1$ the spaces $N(\Lambda_i)$ are naturally modules for $N(\Lambda_0)$ (see Theorem 6.2.21 of [LL]).

If $\langle \Lambda_i, \mathbf{k} \rangle > 1$, then $e^{\lambda_i} \in V_P$ is not a highest weight vector for $\hat{\mathfrak{g}}$, but it is a highest weight vector for $\bar{\mathfrak{n}}$ in the sense that it is annihilated by $\bar{\mathfrak{n}}_+$ and its span is preserved by $\mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d$. We shall use the notation

$$v_{\lambda_i} = e^{\lambda_i} \quad (2.31)$$

for $i = 0, \dots, l$, so that v_{λ_i} agrees with the highest weight vector v_{Λ_i} if $\langle \Lambda_i, \mathbf{k} \rangle = 1$. We now generalize the notion of principal subspace as follows: We define

$$W(\lambda_i) = U(\bar{\mathfrak{n}}) \cdot v_{\lambda_i} \subset V_P \quad (2.32)$$

for each $i = 0, \dots, l$, and we call these the *principal-like subspaces*. More generally, for any $\lambda \in P$ we have the *principal-like subspace*

$$W(\lambda) = U(\bar{\mathfrak{n}}) \cdot e^{\lambda} \subset V_P. \quad (2.33)$$

We also have the *principal-like subspace*

$$W^N(\lambda_i) = U(\bar{\mathfrak{n}}) \otimes_{U(\bar{\mathfrak{n}}_+)} \mathbb{C}v_{\lambda_i}^N \simeq U(\bar{\mathfrak{n}}_-) \cdot v_{\lambda_i}^N, \quad (2.34)$$

where $v_{\lambda_i}^N$ is a highest weight vector for $\bar{\mathfrak{n}}$ and $i = 0, \dots, l$; if $\langle \Lambda_i, \mathbf{k} \rangle = 1$ we take $v_{\lambda_i}^N$ to be the vector $v_{\Lambda_i}^N$ used in (2.11). We view $W^N(\Lambda_i)$ as an $\bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathbb{C}\mathbf{k} \oplus \mathbb{C}d$ -module, as in (2.11), where \mathfrak{h} , \mathbf{k} and d act on $v_{\Lambda_i}^N$ as they do on e^{λ_i} .

Remark 2.2. For each fundamental weight Λ_i with $\langle \Lambda_i, \mathbf{k} \rangle = 1$ the principal-like subspace $W(\lambda_i)$ agrees with the principal subspace $W(\Lambda_i)$, and $W^N(\lambda_i)$ agrees with $W^N(\Lambda_i)$.

Standard arguments show that $W(\lambda_0)$ is a vertex subalgebra of $L(\Lambda_0)$ and that each $W(\lambda_i)$, $i = 0, \dots, l$, is a module for this vertex algebra. Moreover, each $W(\lambda_i)$ is preserved by $L(0)$ and by the action of \mathfrak{h} .

Generalizing (2.8), (2.13) and (2.15) we have the natural maps

$$\begin{aligned} f_{\lambda_i} : U(\bar{\mathfrak{n}}) &\longrightarrow W(\lambda_i), \\ a &\longmapsto a \cdot v_{\lambda_i}, \end{aligned} \quad (2.35)$$

$$\begin{aligned} f_{\lambda_i}^N : U(\bar{\mathfrak{n}}) &\longrightarrow W^N(\lambda_i), \\ a &\longmapsto a \cdot v_{\lambda_i}^N, \end{aligned} \quad (2.36)$$

$$\begin{aligned}\pi_{\lambda_i} : W^N(\lambda_i) &\longrightarrow W(\lambda_i), \\ a \cdot v_{\lambda_i}^N &\longmapsto a \cdot v_{\lambda_i},\end{aligned}\tag{2.37}$$

where $a \in U(\bar{\mathfrak{n}})$, for $i = 0, \dots, l$.

Remark 2.3. Note that the maps f_{λ_i} , $f_{\lambda_i}^N$ and π_{λ_i} indeed agree with the maps f_{Λ_i} , $f_{\Lambda_i}^N$ and π_{Λ_i} , respectively, when $\langle \Lambda_i, \mathbf{k} \rangle = 1$.

Recall from formula (5.1.5), Remark 5.4.2 and Proposition 5.4.7 of [FHL] the (nonzero) intertwining operator

$$\mathcal{Y}(\cdot, x) : V_P \longrightarrow \text{Hom}(V_Q, V_P)[[x, x^{-1}]]\tag{2.38}$$

defined by

$$\mathcal{Y}(w, x)v = e^{xL(-1)}Y(v, -x)w, \quad w \in V_P, \quad v \in V_Q,\tag{2.39}$$

where $L(-1)$ is the usual Virasoro algebra operator (recall (2.29)). In particular, we have

$$\mathcal{Y}(e^{\lambda_i}, x) : V_Q \longrightarrow V_P((x))\tag{2.40}$$

for $i = 0, \dots, l$. By a standard argument (cf. [CLM1]),

$$[x_\alpha(m), \mathcal{Y}(e^{\lambda_i}, x)] = 0$$

for all i . We denote the constant term (the coefficient of x^0) of $\mathcal{Y}(e^{\lambda_i}, x)$ by $\mathcal{Y}_c(e^{\lambda_i}, x)$. Then we have a surjection

$$\mathcal{Y}_c(e^{\lambda_i}, x) : W(\lambda_0) \longrightarrow W(\lambda_i),\tag{2.41}$$

since this map sends v_{λ_0} to v_{λ_i} and $\mathcal{Y}_c(e^{\lambda_i}, x)$ commutes with the action of $\bar{\mathfrak{n}}$. Thus

$$\text{Ker } f_{\lambda_0} \subset \text{Ker } f_{\lambda_i}.\tag{2.42}$$

Indeed, let $a \in U(\bar{\mathfrak{n}})$ such that $a \in \text{Ker } f_{\lambda_0}$; then $a \cdot e^{\lambda_0} = 0$. By applying the map (2.41) and using its properties we have $a \cdot e^{\lambda_i} = 0$, and thus $a \in \text{Ker } f_{\lambda_i}$.

Remark 2.4. The V_Q -module V_P has a structure of an abelian intertwining algebra, as defined and described in Chapter 12 of [DL]. The intertwining operators (2.38) and (2.40) are operators in this abelian intertwining algebra, or more precisely, the restrictions of these intertwining operators to individual sectors corresponding to the cosets of Q in P are the operators that are part of the abelian intertwining algebra, modulo certain normalizations.

The vector space V_P has a natural grading defined by the action of the standard Virasoro algebra operator $L(0)$ introduced earlier, referred to as the grading by *weight*. In particular, we have

$$\text{wt } e^\lambda = \frac{1}{2} \langle \lambda, \lambda \rangle$$

for any $\lambda \in P$, and

$$\text{wt } x_\alpha(m) = -m$$

for any $\alpha \in \Delta$ and $m \in \mathbb{Z}$. The $L(0)$ -eigenspaces of V_p coincide with the eigenspaces for the negative $-d$ of the degree operator d (with the same respective eigenvalues). There are also l gradings by charge on V_p , given by the eigenvalues of the operators $\lambda_i = \lambda_i(0)$, $i = 1, \dots, l$. We will refer to these as the λ_i -charge gradings. The weight and charge gradings are compatible. For any $m \in \mathbb{Z}$, $x_{\alpha_i}(m)$, viewed as either an operator or as an element of $U(\bar{n})$, has weight $-m$ and charge δ_{ij} with respect to λ_j . Also, e^{λ_i} has charge $\langle \lambda_j, \lambda_i \rangle$ with respect to λ_j . We define *total charge* as the sum of the λ_i -charges. We restrict these gradings to the principal-like subspaces $W(\lambda_i)$ for $i = 0, \dots, l$, and in particular to the principal subspaces $W(\Lambda_i)$, and we give the spaces $W^N(\lambda_i)$ and $W^N(\Lambda_i)$ the analogous gradings.

Remark 2.5. Just as in [CalLM1], [CalLM2], we have that $\text{Ker } f_{\Lambda_i}$ and $\text{Ker } \pi_{\Lambda_i}$ are graded by weight and by λ_j -charge for $j = 1, \dots, l$, and these gradings are compatible. More generally, this assertion holds for $\text{Ker } f_{\lambda_i}$ and $\text{Ker } \pi_{\lambda_i}$ for each $i = 0, \dots, l$.

3. Ideals and morphisms

We consider the following formal infinite sums of operators:

$$R_t^i = \sum_{m_1+m_2=-t} x_{\alpha_i}(m_1)x_{\alpha_i}(m_2), \quad t \in \mathbb{Z}, i = 1, \dots, l; \quad (3.1)$$

each R_t^i acts naturally on any highest weight $\hat{\mathfrak{g}}$ -module and more generally on any \bar{n} -module on which the formal sum terminates when applied to any vector. We truncate R_t^i as follows:

$$R_{-1,t}^i = \sum_{m_1, m_2 \leq -1, m_1+m_2=-t} x_{\alpha_i}(m_1)x_{\alpha_i}(m_2) \quad (3.2)$$

for $t \in \mathbb{Z}$ (so that $R_{-1,t}^i = 0$ unless $t \geq 2$) and $i = 1, \dots, l$. When $\mathfrak{g} = \mathfrak{sl}(2)$, and thus only $i = 1$ is relevant, these are the formal sums introduced in [CalLM1] and denoted by R_t^0 . We shall often be viewing $R_{-1,t}^i$ as an element of $U(\bar{n})$, and in fact of $U(\bar{n}_-)$, rather than as an endomorphism of a $\hat{\mathfrak{g}}$ -module. Note that R_t^i and $R_{-1,t}^i$ have weight t , charge 2 with respect to λ_i , and charge 0 with respect to λ_j for $j \neq i$. In this paper we will give two statements for presentations of the principal subspaces $W(\Lambda_i)$, where Λ_i is a fundamental weight such that $\langle \Lambda_i, \mathbf{k} \rangle = 1$, and, more generally, of the principal-like subspaces $W(\lambda_i)$, $0 \leq i \leq l$. One statement involves left ideals of $U(\bar{n})$ generated by elements of type (3.2), while the other statement uses two-sided ideals of a certain completion of $U(\bar{n})$, ideals generated by the formal infinite sums (3.1).

Let J be the left ideal of $U(\bar{n})$ generated by the elements $R_{-1,t}^i$ for $t \geq 2$ and $i = 1, \dots, l$:

$$J = \sum_{i=1}^l \sum_{t \geq 2} U(\bar{n}) R_{-1,t}^i. \quad (3.3)$$

By analogy with the corresponding constructions in [CalLM1], [CalLM2] (which involved $U(\bar{n}_-)$), we set

$$I_{\lambda_0} = J + U(\bar{n})\bar{n}_+ \subset U(\bar{n}) \quad (3.4)$$

and

$$I_{\lambda_i} = J + U(\bar{n})\bar{n}_+ + U(\bar{n})x_{\alpha_i}(-1) = I_{\lambda_0} + U(\bar{n})x_{\alpha_i}(-1) \subset U(\bar{n}), \quad i = 1, \dots, l. \quad (3.5)$$

Remark 3.1. Since in this paper we are concerned with more general structures than the principal subspaces of the level one standard $\hat{\mathfrak{g}}$ -modules, and since these structures, $W(\lambda_i)$, are indexed by the fundamental weights of \mathfrak{g} rather than the fundamental weights of $\hat{\mathfrak{g}}$, we use the notation I_{λ_i} rather than I_{Λ_i} . Note that the ideals I_{λ_0} and I_{λ_1} are the same as I_{Λ_0} and I_{Λ_1} in [CalLM1] modulo $U(\bar{n})\bar{n}_+$.

Remark 3.2. The left ideals I_{λ_i} , $i = 0, \dots, l$, are graded by weight and by λ_j -charge for $j = 1, \dots, l$, and these gradings are compatible.

For any $\lambda \in P$ and character $\nu: Q \rightarrow \mathbb{C}^*$, we define a map $\tau_{\lambda, \nu}$ on \bar{n} by

$$\tau_{\lambda, \nu}(x_\alpha(m)) = \nu(\alpha)x_\alpha(m - \langle \lambda, \alpha \rangle)$$

for $\alpha \in \Delta_+$ and $m \in \mathbb{Z}$. It is easy to see that $\tau_{\lambda, \nu}$ is an automorphism of \bar{n} . We will distinguish an important special case when ν is trivial (i.e., $\nu = 1$): We set

$$\tau_\lambda = \tau_{\lambda, 1}.$$

The map $\tau_{\lambda, \nu}$ extends canonically to an automorphism of $U(\bar{n})$, which we also denote by $\tau_{\lambda, \nu}$, so that

$$\tau_{\lambda, \nu}(x_{\beta_1}(m_1) \cdots x_{\beta_k}(m_k)) = \nu(\beta_1 + \cdots + \beta_k)x_{\beta_1}(m_1 - \langle \lambda, \beta_1 \rangle) \cdots x_{\beta_k}(m_k - \langle \lambda, \beta_k \rangle) \quad (3.6)$$

for $\beta_1, \dots, \beta_k \in \Delta_+$ and $m_1, \dots, m_k \in \mathbb{Z}$.

Such automorphisms (or translations) generalize certain shift maps that appeared in our previous work [CalLM1], [CalLM2] (recall [CalLM1], (3.20) and [CalLM2], (4.27)). Notice that for any $\lambda, \mu \in P$ and any characters ν and ν' on Q , we have

$$\tau_{\lambda, \nu} \tau_{\mu, \nu'} = \tau_{\lambda + \mu, \nu \nu'} \quad \text{and} \quad \tau_{\lambda, \nu}^{-1} = \tau_{-\lambda, \nu^{-1}}. \quad (3.7)$$

Recall the (multiplicative) commutator map $c(\cdot, \cdot)$ on $P \times P$, satisfying $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}$ for $\alpha, \beta \in Q$. For $\lambda \in P$, the map

$$c_\lambda(\alpha) = c(\alpha, \lambda), \quad \alpha \in Q, \quad (3.8)$$

is a character on Q .

Remark 3.3. Assume that $a \in U(\bar{n})$ is a nonzero element homogeneous with respect to the weight and λ_i -charge gradings. For any $\lambda \in P$ and character ν on Q , $\tau_{\lambda, \nu}(a)$ and $\tau_{\lambda, \nu}^{-1}(a)$ are also homogeneous and have the same λ_i -charge n_i as a . Moreover,

$$\text{wt } \tau_{\lambda_i, \nu}(a) = \text{wt } a + n_i, \quad (3.9)$$

$$\text{wt } \tau_{\lambda_i, \nu}^{-1}(a) = \text{wt } a - n_i. \quad (3.10)$$

In particular, if $n_i > 0$ then

$$\text{wt } \tau_{\lambda_i, \nu}(a) > \text{wt } a, \quad (3.11)$$

$$\text{wt } \tau_{\lambda_i, \nu}^{-1}(a) < \text{wt } a. \quad (3.12)$$

Lemma 3.1. For every $i = 1, \dots, l$ and character ν we have

$$\tau_{\lambda_i, \nu}(I_{\lambda_0}) \subset I_{\lambda_i}. \quad (3.13)$$

Proof. Because I_{λ_0} is a homogeneous ideal, it is sufficient to consider τ_{λ_i} . By (3.2) and (3.6), for each $t \geq 2$ we have

$$\tau_{\lambda_i}(R_{-1,t}^i) = R_{-1,t+2}^i + ax_{\alpha_i}(-1),$$

where $a \in U(\bar{\mathfrak{n}})$, and

$$\tau_{\lambda_i}(R_{-1,t}^j) = R_{-1,t}^j \quad \text{for } j \neq i.$$

Since J is the left ideal of $U(\bar{\mathfrak{n}})$ generated by the $R_{-1,t}^i$ for $t \geq 2$ and $i = 1, \dots, l$, we have

$$\tau_{\lambda_i}(J) \subset I_{\lambda_i}.$$

Note that for any $\beta \in \Delta_+$ and $m \geq 0$, $x_\beta(m)$ can be expressed as a linear combination of monomials of the form $x_{\alpha_{r_1}}(m_1) \cdots x_{\alpha_{r_k}}(m_k)$ such that $m_k \geq 0$ and $\alpha_{r_1}, \dots, \alpha_{r_k}$ are simple roots. If $r_k \neq i$ we have

$$\tau_{\lambda_i}(x_{\alpha_{r_1}}(m_1) \cdots x_{\alpha_{r_k}}(m_k)) \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+$$

and if $r_k = i$ we have

$$\tau_{\lambda_i}(x_{\alpha_{r_1}}(m_1) \cdots x_{\alpha_{r_k}}(m_k)) \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+ + U(\bar{\mathfrak{n}})x_{\alpha_i}(-1),$$

whether $m_k > 0$ or $m_k = 0$. Thus

$$\tau_{\lambda_i}(U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}_+) \subset I_{\lambda_i}.$$

This completes the proof of (3.13). \square

Consider the weights

$$\omega_i = \alpha_i - \lambda_i \in P \quad (3.14)$$

and the automorphisms τ_{ω_i} of $U(\bar{\mathfrak{n}})$ for $i = 1, \dots, l$. These weights and certain maps associated with them played an important role in [Cal1] (see [Cal1], (4.12)). For any character ν on Q , define the linear map

$$\begin{aligned} \sigma_{\omega_i, \nu} : U(\bar{\mathfrak{n}}) &\longrightarrow U(\bar{\mathfrak{n}}), \\ a &\longmapsto \tau_{\omega_i, \nu}(a)x_{\alpha_i}(-1). \end{aligned}$$

This map is injective. To simplify the notation, we will write

$$\sigma_{\omega_i} = \sigma_{\omega_i, 1}.$$

Lemma 3.2. For every $i = 1, \dots, l$ and character ν we have

$$\sigma_{\omega_i, \nu}(I_{\lambda_i}) \subset I_{\lambda_0}. \quad (3.15)$$

Proof. Because the ideal I_{λ_i} is generated by homogeneous elements, as in the previous lemma we may assume that ν is trivial. Let $a \in U(\bar{n})$. Then

$$\sigma_{\omega_i}(ax_{\alpha_i}(-1)) = \tau_{\omega_i}(a)x_{\alpha_i}(-2)x_{\alpha_i}(-1) \in J \subset I_{\lambda_0}.$$

Thus we have

$$\sigma_{\omega_i}(U(\bar{n})x_{\alpha_i}(-1)) \subset I_{\lambda_0}. \quad (3.16)$$

Next we show that

$$\sigma_{\omega_i}(x_{\beta}(m)) \in I_{\lambda_0} \quad (3.17)$$

for any $\beta \in \Delta_+$ and $m \geq 0$. As in Lemma 3.1 it is enough to show that (3.17) holds for $x_{\alpha_j}(m)$, where α_j is a simple root and $m \geq 0$. We shall use the fact that for any $\alpha, \beta \in \Delta$ such that $\alpha + \beta \in \Delta$ we have $[x_{\alpha}, x_{\beta}] = C_{\alpha, \beta}x_{\alpha+\beta}$, where $C_{\alpha, \beta} \neq 0$. For any $m \in \mathbb{Z}$ we have

$$\sigma_{\omega_i}(x_{\alpha_j}(m)) = \begin{cases} x_{\alpha_i}(m-1)x_{\alpha_i}(-1) & \text{if } i = j, \\ x_{\alpha_j}(m+1)x_{\alpha_i}(-1) & \text{if } i \neq j \text{ and } a_{ij} = -1, \\ x_{\alpha_j}(m)x_{\alpha_i}(-1) & \text{if } i \neq j \text{ and } a_{ij} = 0. \end{cases} \quad (3.18)$$

For $m \geq 0$,

$$x_{\alpha_i}(m-1)x_{\alpha_i}(-1) = x_{\alpha_i}(-1)x_{\alpha_i}(m-1) \in I_{\lambda_0},$$

$$x_{\alpha_j}(m+1)x_{\alpha_i}(-1) = C_{\alpha_j, \alpha_i}x_{\alpha_i+\alpha_j}(m) + x_{\alpha_i}(-1)x_{\alpha_j}(m+1) \in I_{\lambda_0} \quad \text{if } i \neq j, a_{ij} = -1,$$

$$x_{\alpha_j}(m)x_{\alpha_i}(-1) = x_{\alpha_i}(-1)x_{\alpha_j}(m) \in I_{\lambda_0} \quad \text{if } i \neq j, a_{ij} = 0,$$

and so $\sigma_{\omega_i}(x_{\alpha_j}(m)) \in I_{\lambda_0}$. Thus (3.17) holds and we have

$$\sigma_{\omega_i}(U(\bar{n})\bar{n}_+) \subset I_{\lambda_0}. \quad (3.19)$$

We now show that

$$\sigma_{\omega_i}(J) \subset I_{\lambda_0}. \quad (3.20)$$

Since J is the left ideal of $U(\bar{n})$ generated by $R_{-1,t}^j$ for $t \in \mathbb{Z}$ and $j = 1, \dots, l$, it is sufficient to show that

$$\sigma_{\omega_i}(R_{-1,t}^j) \in I_{\lambda_0} \quad \text{for } t \in \mathbb{Z}, j = 1, \dots, l.$$

For $i = j$, we have

$$\sigma_{\omega_i}(R_{-1,t}^i) = x_{\alpha_i}(-1)R_{-1,t+2}^i + ax_{\alpha_i}(-1)^2 \in I_{\lambda_0}, \quad (3.21)$$

where $a \in U(\bar{n})$.

If $i \neq j$ and $a_{ij} = 0$, we have

$$\sigma_{\omega_i}(R_{-1,t}^j) = R_{-1,t}^j x_{\alpha_i}(-1) = x_{\alpha_i}(-1)R_{-1,t}^j \in I_{\lambda_0}. \quad (3.22)$$

If $i \neq j$ and $a_{ij} = -1$, then $\alpha_i + \alpha_j \in \Delta_+$ and $2\alpha_i + \alpha_j \notin \Delta$, and thus $[x_{\alpha_i}(m), x_{\alpha_i+\alpha_j}(n)] = 0$ for any $m, n \in \mathbb{Z}$. Then for any $t \in \mathbb{Z}$ we have

$$\begin{aligned} \sigma_{\omega_i}(R_{-1,t}^j) &= \sum_{m_1, m_2 \leq -1, m_1+m_2=-t} x_{\alpha_j}(m_1+1)x_{\alpha_j}(m_2+1)x_{\alpha_i}(-1) \\ &= \sum_{m_1, m_2 \leq -1, m_1+m_2=-t} C_{\alpha_j, \alpha_i} x_{\alpha_j}(m_1+1)x_{\alpha_i+\alpha_j}(m_2) \end{aligned} \quad (3.23)$$

$$+ \sum_{m_1, m_2 \leq -1, m_1+m_2=-t} C_{\alpha_j, \alpha_i} x_{\alpha_i+\alpha_j}(m_1)x_{\alpha_j}(m_2+1) \quad (3.24)$$

$$+ x_{\alpha_i}(-1) \sum_{m_1, m_2 \leq -1, m_1+m_2=-t} x_{\alpha_j}(m_1+1)x_{\alpha_j}(m_2+1). \quad (3.25)$$

The last term on the right-hand side is of the form

$$x_{\alpha_i}(-1)R_{-1,t-2}^j + a, \quad \text{where } a \in U(\bar{n})\bar{n}_+ \quad (3.26)$$

(and this is true even if $t = 2$ or $t = 3$) and thus this term is in I_{λ_0} . We rewrite (3.23) as follows:

$$\begin{aligned} &\sum_{m_1, m_2 \leq -1, m_1+m_2=-t} C_{\alpha_j, \alpha_i} x_{\alpha_j}(m_1+1)x_{\alpha_i+\alpha_j}(m_2) \\ &= C_{\alpha_j, \alpha_i} x_{\alpha_j}(0)x_{\alpha_i+\alpha_j}(-t+1) + \sum_{m_1, m_2 \leq -1, m_1+m_2=-t+1} C_{\alpha_j, \alpha_i} x_{\alpha_j}(m_1)x_{\alpha_i+\alpha_j}(m_2). \end{aligned} \quad (3.27)$$

Similarly,

$$\begin{aligned} &\sum_{m_1, m_2 \leq -1, m_1+m_2=-t} C_{\alpha_j, \alpha_i} x_{\alpha_i+\alpha_j}(m_1)x_{\alpha_j}(m_2+1) \\ &= C_{\alpha_j, \alpha_i} x_{\alpha_i+\alpha_j}(-t+1)x_{\alpha_j}(0) + \sum_{m_1, m_2 \leq -1, m_1+m_2=-t+1} C_{\alpha_j, \alpha_i} x_{\alpha_i+\alpha_j}(m_1)x_{\alpha_j}(m_2). \end{aligned} \quad (3.28)$$

Notice that

$$[R_{-1,t-1}^j, x_{\alpha_i}(0)] = R_{-1,t-1}^j x_{\alpha_i}(0) - x_{\alpha_i}(0)R_{-1,t-1}^j \in J + U(\bar{n})\bar{n}_+ = I_{\lambda_0} \quad (3.29)$$

and that, on the other hand,

$$\begin{aligned} [R_{-1,t-1}^j, x_{\alpha_i}(0)] &= \sum_{m_1, m_2 \leq -1, m_1+m_2=-t+1} C_{\alpha_j, \alpha_i} x_{\alpha_j}(m_1)x_{\alpha_i+\alpha_j}(m_2) \\ &+ \sum_{m_1, m_2 \leq -1, m_1+m_2=-t+1} C_{\alpha_j, \alpha_i} x_{\alpha_i+\alpha_j}(m_1)x_{\alpha_j}(m_2). \end{aligned} \quad (3.30)$$

Combining (3.26)–(3.30) we obtain, for any $t \in \mathbb{Z}$ (in particular, for $t \geq 2$),

$$\sigma_{\omega_i}(R_{-1,t}^j) \in I_{\lambda_0} \quad \text{if } i \neq j \text{ and } a_{ij} = -1. \quad (3.31)$$

By (3.21), (3.22) and (3.31) we obtain (3.20). Combining (3.16), (3.19) and (3.20) we obtain the inclusion (3.15). \square

Consider the composition $\sigma_{\omega_i} \tau_{\lambda_i}$. As a consequence of Lemmas 3.1 and 3.2 we obtain:

Corollary 3.1. *For every $i = 1, \dots, l$ and characters ν and ν' on Q we have*

$$\sigma_{\omega_i, \nu} \tau_{\lambda_i, \nu'}(I_{\lambda_0}) \subset I_{\lambda_0}. \quad (3.32)$$

For any $\lambda \in P$ recall the linear isomorphism

$$e_\lambda : V_P \longrightarrow V_P.$$

In particular, since

$$e_{\lambda_i} \cdot v_{\lambda_0} = v_{\lambda_i}$$

(recall (2.31) and (2.20)), we have linear isomorphisms

$$e_{\lambda_i} : W(\lambda_0) \longrightarrow W(\lambda_i) \quad \text{for } i = 1, \dots, l, \quad (3.33)$$

given explicitly as follows: Since

$$e_{\lambda_i} x_\alpha(m) = c(\alpha, -\lambda_i) x_\alpha(m - \langle \alpha, \lambda_i \rangle) e_{\lambda_i} \quad \text{for } \alpha \in \Delta_+ \text{ and } m \in \mathbb{Z}$$

(recall (2.28)), we have

$$e_{\lambda_i}(a \cdot v_{\lambda_0}) = \tau_{\lambda_i, c_{-\lambda_i}}(a) \cdot v_{\lambda_i}, \quad a \in U(\bar{\mathfrak{n}}). \quad (3.34)$$

Generalizing the corresponding constructions in [CalLM1], [CalLM2], we construct a linear lifting

$$\widehat{e_{\lambda_i}} : W^N(\lambda_0) \longrightarrow W^N(\lambda_i) \quad (3.35)$$

of each map (3.33), in the sense that the diagram

$$\begin{array}{ccc} W^N(\lambda_0) & \xrightarrow{\widehat{e_{\lambda_i}}} & W^N(\lambda_i) \\ \pi_{\lambda_0} \downarrow & & \downarrow \pi_{\lambda_i} \\ W(\lambda_0) & \xrightarrow{e_{\lambda_i}} & W(\lambda_i) \end{array}$$

will commute, for $i = 1, \dots, l$, by taking

$$\widehat{e_{\lambda_i}}(a \cdot v_{\lambda_0}^N) = \tau_{\lambda_i, c_{-\lambda_i}}(a) \cdot v_{\lambda_i}^N \quad \text{for } a \in U(\bar{\mathfrak{n}}_-); \quad (3.36)$$

the map $\widehat{e_{\lambda_i}}$ is well defined since $W^N(\lambda_0)$ is a free $U(\bar{\mathfrak{n}}_-)$ -module.

We have:

Corollary 3.2. *For $i = 1, \dots, l$,*

$$\widehat{e_{\lambda_i}}(I_{\lambda_0} \cdot v_{\lambda_0}^N) \subset I_{\lambda_i} \cdot v_{\lambda_i}^N. \quad (3.37)$$

Proof. Let $a \in I_{\lambda_0}$. We may assume that $a \in U(\bar{n}_-)$. Indeed, since $a \in U(\bar{n})$ we can write $a = b + c$, where $b \in U(\bar{n})\bar{n}_+$ and $c \in U(\bar{n}_-)$ (recall (2.3)), and since $b \in I_{\lambda_0}$, $c \in I_{\lambda_0}$ as well. Now by applying (3.36) and Lemma 3.1 to the element $a \in I_{\lambda_0} \cap U(\bar{n}_-)$ we obtain the inclusion (3.37). \square

Recall the weights $\omega_i = \alpha_i - \lambda_i$, $i = 1, \dots, l$. We now consider the linear isomorphism

$$e_{\omega_i} : V_P \longrightarrow V_P \quad (3.38)$$

and its restriction to the principal-like subspace $W(\lambda_i)$. Since, by (2.28),

$$e_{\omega_i} x_\alpha(m) = c(\alpha, -\omega_i) x_\alpha(m - \langle \alpha, \omega_i \rangle) e_{\omega_i} \quad \text{for } \alpha \in \Delta_+ \text{ and } m \in \mathbb{Z}$$

and

$$e_{\omega_i} \cdot v_{\lambda_i} = e_{\omega_i} \cdot e^{\lambda_i} = \epsilon(\omega_i, \lambda_i) e^{\alpha_i} = \epsilon(\omega_i, \lambda_i) x_{\alpha_i}(-1) \cdot v_{\lambda_0},$$

we have

$$e_{\omega_i}(a \cdot v_{\lambda_i}) = \epsilon(\omega_i, \lambda_i) \tau_{\omega_i, c-\omega_i}(a) x_{\alpha_i}(-1) \cdot v_{\lambda_0} = \epsilon(\omega_i, \lambda_i) \sigma_{\omega_i, c-\omega_i}(a) \cdot v_{\lambda_0} \quad (3.39)$$

for $a \in U(\bar{n})$, giving us a linear injection

$$e_{\omega_i} : W(\lambda_i) \longrightarrow W(\lambda_0) \quad (3.40)$$

for $i = 1, \dots, l$.

4. Presentations of the principal subspaces

In this section we prove natural presentations of the principal subspaces of the level one standard modules for $\hat{\mathfrak{g}}$, where \mathfrak{g} is of type A_l , $l \geq 1$, D_l , $l \geq 4$, E_6 , E_7 and E_8 (Theorem 4.2). More generally, we prove natural presentations of the principal-like subspaces $W(\lambda_i) \subset V_P$ for $i = 0, \dots, l$ (Theorem 4.1). In order to obtain these results we generalize the proof of Theorem 2.2 (which is equivalent to Theorem 2.1) of [CalLM1]. Following [Cal1], we give a reformulation of the presentations of the principal subspaces, and more generally, of the principal-like subspaces, in terms of ideals of a certain completion of the universal enveloping algebra $U(\bar{n})$ (Theorem 4.3), and we show that Theorem 4.1 implies Theorem 4.3.

First we state:

Theorem 4.1. *For every $i = 0, \dots, l$, we have*

$$\text{Ker } f_{\lambda_i} = I_{\lambda_i}, \quad (4.1)$$

or equivalently,

$$\text{Ker } \pi_{\lambda_i} = I_{\lambda_i} \cdot v_{\lambda_i}^N. \quad (4.2)$$

(To see that (4.2) implies that $\text{Ker } f_{\lambda_i} \subset I_{\lambda_i}$, we use the direct sum decomposition (2.3).)

Recall that when $\langle \Lambda_i, \mathbf{k} \rangle = 1$, the principal-like subspace $W(\lambda_i)$ agrees with the principal subspace $W(\Lambda_i)$ of the level one standard module $L(\Lambda_i)$, and f_{λ_i} and π_{λ_i} agree with the maps f_{Λ_i} and π_{Λ_i} , respectively (cf. Remarks 2.2 and 2.3). As a particular case of Theorem 4.1 we have the following presentations of the principal subspaces $W(\Lambda_i)$:

Theorem 4.2. For any fundamental weight Λ_i such that $\langle \Lambda_i, \mathbf{k} \rangle = 1$,

$$\text{Ker } f_{\Lambda_i} = I_{\lambda_i}, \quad (4.3)$$

or equivalently,

$$\text{Ker } \pi_{\Lambda_i} = I_{\lambda_i} \cdot v_{\Lambda_i}^N. \quad (4.4)$$

Proof of Theorem 4.1. For each $j = 1, \dots, l$, the square of the vertex operator $Y(e^{\alpha_j}, x)$ is well defined and equals zero on V_p and hence on each $W(\lambda_i)$. Thus $Y(e^{\alpha_j}, x)^2 = \sum_{t \in \mathbb{Z}} R_t^j x^{t-2}$ equals zero on $W(\lambda_i)$ (recall (3.1)), so that $R_t^j = 0$ on $W(\lambda_i)$ for each $t \in \mathbb{Z}$. This combined with the highest weight vector property of v_{λ_i} and the fact that

$$x_{\alpha_i}(-1) \cdot v_{\lambda_i} = 0 \quad \text{if } i > 0$$

(which follows from (2.24)) implies that

$$I_{\lambda_i} \subset \text{Ker } f_{\lambda_i},$$

and so

$$I_{\lambda_i} \cdot v_{\lambda_i}^N \subset \text{Ker } \pi_{\lambda_i}$$

for $i = 0, \dots, l$. We will now prove the inclusions

$$\text{Ker } \pi_{\lambda_i} \subset I_{\lambda_i} \cdot v_{\lambda_i}^N \quad \text{for } i = 0, \dots, l. \quad (4.5)$$

We first claim that (4.2) with $i = 1, \dots, l$ follows from (4.2) with $i = 0$, whose truth we now assume. Let $i > 0$ and let $a \cdot v_{\lambda_i}^N \in \text{Ker } \pi_{\lambda_i}$, where $a \in U(\bar{\mathfrak{n}})$. Then $a \cdot v_{\lambda_i} = 0$ in $W(\lambda_i)$, and so

$$\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0} = 0$$

in $W(\lambda_0)$, by (3.34). Thus

$$\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0}^N \in \text{Ker } \pi_{\lambda_0} = I_{\lambda_0} \cdot v_{\lambda_0}^N.$$

Now by Corollary 3.2 we have

$$\widehat{e}_{\lambda_i}(\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0}^N) \in I_{\lambda_i} \cdot v_{\lambda_i}^N,$$

and from (3.36) we get

$$a \cdot v_{\lambda_i}^N \in I_{\lambda_i} \cdot v_{\lambda_i}^N \quad \text{for } i = 1, \dots, l.$$

This proves our claim.

As in [CalLM1], [CalLM2] we will use a contradiction argument to prove the inclusion

$$\text{Ker } \pi_{\lambda_0} \subset I_{\lambda_0} \cdot v_{\lambda_0}^N, \quad (4.6)$$

which is all that remains to prove. Suppose then that there exists $a \in U(\bar{\mathfrak{n}})$ such that

$$a \cdot v_{\lambda_0}^N \in \text{Ker } \pi_{\lambda_0} \quad \text{but} \quad a \cdot v_{\lambda_0}^N \notin I_{\lambda_0} \cdot v_{\lambda_0}^N. \quad (4.7)$$

We may and do assume that a is homogeneous with respect to the weight and λ_i -charge gradings (recall Remarks 2.5 and 3.2). The element a is certainly nonzero, and it is also nonconstant because otherwise, $a \cdot v_{\lambda_0}^N \notin \text{Ker } \pi_{\lambda_0}$. Using this and the decomposition (2.3), we further see that a has positive weight, since otherwise, a would be an element of $U(\bar{n})\bar{n}_+$ and hence of I_{λ_0} . Denote by n the total charge of a , namely, $n = n_1 + \dots + n_l$, where $n_i \geq 0$ is the λ_i -charge of a for $i = 1, \dots, l$. Note that $n > 0$; otherwise, a is constant. Take n to be the minimum total charge for all the homogeneous elements a satisfying (4.7). Among all the homogeneous elements of total charge n satisfying (4.7), we choose a to be an element of the smallest possible (necessarily positive) weight. Fix any index i for which $n_i > 0$.

We claim that $a \in I_{\lambda_i}$. Assume then that

$$a \notin I_{\lambda_i}. \quad (4.8)$$

Since $a \cdot v_{\lambda_0}^N \in \text{Ker } \pi_{\lambda_0}$, $a \in \text{Ker } f_{\lambda_0} \subset \text{Ker } f_{\lambda_i}$ (see (2.42)). Then $a \cdot v_{\lambda_i} = 0$ in $W(\lambda_i)$, and so

$$e_{\lambda_i}(\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0}) = a \cdot v_{\lambda_i} = 0,$$

by (3.34). This implies that $\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0} = 0$, or equivalently, that

$$\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0}^N \in \text{Ker } \pi_{\lambda_0}. \quad (4.9)$$

Also,

$$\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \cdot v_{\lambda_0}^N \notin I_{\lambda_0} \cdot v_{\lambda_0}^N; \quad (4.10)$$

otherwise, we would have $\tau_{\lambda_i, c-\lambda_i}^{-1}(a) \in I_{\lambda_0}$ (from (2.3)) and so by Lemma 3.1 we would get $a \in I_{\lambda_i}$, contradicting (4.8). The elements $\tau_{\lambda_i, c-\lambda_i}^{-1}(a)$ and a have the same λ_j -charge for each j , and hence the same total charge n , and

$$\text{wt } \tau_{\lambda_i, c-\lambda_i}^{-1}(a) < \text{wt } a, \quad (4.11)$$

by Remark 3.3. Now (4.9), (4.10) and (4.11) contradict our choice of a and thus we get

$$a \in I_{\lambda_i}. \quad (4.12)$$

Since $I_{\lambda_i} = I_{\lambda_0} + U(\bar{n})x_{\alpha_i}(-1)$, there exist $b \in I_{\lambda_0}$ and $c \in U(\bar{n})$ such that

$$a = b + cx_{\alpha_i}(-1). \quad (4.13)$$

We may and do assume that b and c are homogeneous with respect to the weight and λ_j -charge gradings. Note that b has the same weight and the same total charge n as a ; the total charge of c is $n-1$; and also, $\text{wt } c = \text{wt } a - 1$.

We now claim that

$$cx_{\alpha_i}(-1) \in I_{\lambda_0}. \quad (4.14)$$

Assume that (4.14) does not hold. Then

$$\tau_{\alpha_i, c-\alpha_i}^{-1}(c) \cdot v_{\lambda_0}^N \notin I_{\lambda_0} \cdot v_{\lambda_0}^N. \quad (4.15)$$

Indeed, from (3.7) we have

$$\tau_{\lambda_i, c-\alpha_i} \tau_{\alpha_i, c-\alpha_i}^{-1} = \tau_{\lambda_i, c-\alpha_i} \tau_{-\alpha_i, (c-\alpha_i)}^{-1} = \tau_{-\omega_i}$$

and

$$\sigma_{\omega_i} \tau_{-\omega_i}(c) = \tau_{\omega_i}(\tau_{-\omega_i}(c)) x_{\alpha_i}(-1) = c x_{\alpha_i}(-1),$$

so that

$$c x_{\alpha_i}(-1) = (\sigma_{\omega_i} \tau_{\lambda_i, c-\alpha_i})(\tau_{\alpha_i, c-\alpha_i}^{-1}(c)),$$

and Corollary 3.1 yields $\tau_{\alpha_i, c-\alpha_i}^{-1}(c) \notin I_{\lambda_0}$; thus (4.15) holds. By (2.28), for $f \in U(\bar{n})$,

$$e_{\alpha_i} f = \tau_{\alpha_i, c-\alpha_i}(f) e_{\alpha_i}$$

as operators, so that

$$e_{\alpha_i}(\tau_{\alpha_i, c-\alpha_i}^{-1}(c) \cdot v_{\lambda_0}) = c e_{\alpha_i} \cdot v_{\lambda_0} = c e^{\alpha_i} = c x_{\alpha_i}(-1) \cdot v_{\lambda_0} = (a - b) \cdot v_{\lambda_0} = 0,$$

and so

$$\tau_{\alpha_i, c-\alpha_i}^{-1}(c) \cdot v_{\lambda_0}^N \in \text{Ker } \pi_{\lambda_0}. \quad (4.16)$$

Also, by Remark 3.3, $\tau_{\alpha_i, c-\alpha_i}^{-1}(c)$ is homogeneous and has the same total charge as c , namely, $n - 1$. Thus (4.15) and (4.16) contradict our choice of the element a , and so we have $c x_{\alpha_i}(-1) \in I_{\lambda_0}$, proving our claim (4.14).

It follows that $a \in I_{\lambda_0}$. This shows that our initial assumption is false, and therefore we have

$$\text{Ker } \pi_{\lambda_0} \subset I_{\lambda_0} \cdot v_{\lambda_0}^N. \quad \square$$

Remark 4.1. We now comment on the similarities between the proofs of Theorem 4.1 in this paper and Theorem 2.2 of [CalLM1], which deals with the $\mathfrak{sl}(2)$ level one case. Note that when $\mathfrak{g} = \mathfrak{sl}(2)$ the principal-like subspaces coincide with the principal subspaces. One difference between these two proofs is that here we use $U(\bar{n})$ rather than $U(\bar{n}_-)$ (which was sufficient for the $\mathfrak{sl}(2)$ case because \bar{n} is abelian), and this simplifies the argument somewhat. In the proof of Theorem 4.1 the claim that $a \in I_{\lambda_i}$ shows that there is in fact a homogeneous element lying in $U(\bar{n})x_{\alpha_i}(-1)$ and in addition having all the properties of a , since $c x_{\alpha_i}(-1)$ has the same weight and charge as the element a and, in addition, it satisfies (4.7). When $\mathfrak{g} = \mathfrak{sl}(2)$ this claim is similar to the one that appears in the proof of Theorem 2.2 of [CalLM1], with $U(\bar{n})$ replaced by $U(\bar{n}_-)$. Also, the proof of (4.14) follows the lines of the last part of the proof of Theorem 2.2 of [CalLM1], except that here our minimal counterexample involves charge as well as weight. In the $\mathfrak{sl}(2)$ case, $\tau_{\alpha_1, c-\alpha_1}^{-1}$ and e_{α_1} , respectively, in this paper are the same as τ^{-2} and $e^{\alpha/2} \circ e^{\alpha/2}$, respectively, in [CalLM1], and so in the $\mathfrak{sl}(2)$ case, it was not necessary to minimize charge as well as weight.

Recall from [Cal1] the two-sided ideal, denoted by \mathcal{J} , of $\widetilde{U(\bar{n})}$, the completion of $U(\bar{n})$ in the sense of [LW3] or [MP1], [MP2], generated by the elements R_t^j for all $t \in \mathbb{Z}$ and $j = 1, \dots, l$, where the R_t^j are defined in (3.1). The decomposition (2.3) implies:

$$\widetilde{U(\bar{n})} = U(\bar{n}_-) + \widetilde{U(\bar{n})\bar{n}_+}. \quad (4.17)$$

We now give a different description of the ideals $\text{Ker } f_{\Lambda_i}$, and more generally, $\text{Ker } f_{\lambda_i}$, in terms of the two-sided ideal \mathcal{J} :

Theorem 4.3. *The annihilator $\text{Ker } f_{\Lambda_0}$ in $U(\bar{n})$ of the highest weight vector of $L(\Lambda_0)$ is described as follows:*

$$\text{Ker } f_{\Lambda_0} \equiv \mathcal{J} \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+}. \quad (4.18)$$

Moreover, for the principal-like subspaces $W(\lambda_i)$, $i = 1, \dots, l$, we have:

$$\text{Ker } f_{\lambda_i} \equiv \mathcal{J} + U(\bar{n})x_{\alpha_i}(-1) \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+}. \quad (4.19)$$

Proof. The inclusions

$$\mathcal{J} \subset \text{Ker } f_{\Lambda_0} \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+}$$

and

$$\mathcal{J} + U(\bar{n})x_{\alpha_i}(-1) \subset \text{Ker } f_{\lambda_i} \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+}$$

follow from the definition of the ideal \mathcal{J} , the first paragraph of the proof of Theorem 4.1, and (4.17). Observe that

$$J \subset \mathcal{J} \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+}. \quad (4.20)$$

Let $a \in \text{Ker } f_{\Lambda_0}$. By Theorem 4.2 and (3.4) we have $a \in J + U(\bar{n})\bar{n}_+$. Then by (4.20) we have $a \in \mathcal{J} \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+}$. This proves the inclusion

$$\text{Ker } f_{\Lambda_0} \subset \mathcal{J} \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+},$$

and thus (4.18). Now let $a \in \text{Ker } f_{\lambda_i}$, $i \neq 0$. Using Theorem 4.1, (3.5) and (4.20) we obtain the inclusion

$$\text{Ker } f_{\lambda_i} \subset \mathcal{J} + U(\bar{n})x_{\alpha_i}(-1) \quad \text{modulo } \widetilde{U(\bar{n})\bar{n}_+},$$

and so formula (4.19) holds. \square

5. q -Difference equations

In this section we will use the presentations proved in Theorems 4.1 and 4.2 in order to construct canonical exact sequences for the principal(-like) subspaces. As a consequence we derive q -difference equations (recursions) for the (multi-)graded dimensions of these subspaces and by solving these equations we obtain explicit formulas for the graded dimensions of the subspaces. Since the proofs of the results in this section use exactly the same arguments as the proofs of the corresponding results in [CLM1], [CLM2] and [Cal1], we will only sketch the proofs here.

Using the injection (3.40) and the surjection (2.41), associated to the relevant intertwining operators, we obtain:

Theorem 5.1. *For every $i = 1, \dots, l$ we have the following short exact sequence of maps among principal-like subspaces:*

$$0 \longrightarrow W(\lambda_i) \xrightarrow{e_{\omega_i}} W(\lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_i}, x)} W(\lambda_i) \longrightarrow 0. \quad (5.1)$$

In particular, for each fundamental weight Λ_i of $\hat{\mathfrak{g}}$ such that $\langle \Lambda_i, \mathbf{k} \rangle = 1$ we have the following short exact sequence among principal subspaces:

$$0 \longrightarrow W(\Lambda_i) \xrightarrow{e_{\omega_i}} W(\Lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_i}, x)} W(\Lambda_i) \longrightarrow 0. \quad (5.2)$$

Proof. The proof follows the same lines as in [CLM1], [CLM2], or more specifically, as in [Cal1]. The main step is the exactness of the chain complex at the middle, and this is where the presentation result, Theorem 4.1 (or Theorem 4.2), is used. In fact, the key steps, easily proved from the above, are:

$$\text{Ker } \mathcal{Y}_c(e^{\lambda_i}, x) = \text{Ker } f_{\lambda_i} \cdot v_{\lambda_0},$$

so that by Theorem 4.1,

$$\text{Ker } \mathcal{Y}_c(e^{\lambda_i}, x) = I_{\lambda_i} \cdot v_{\lambda_0},$$

whereas

$$\text{Im } e_{\omega_i} = U(\bar{\mathfrak{n}})x_{\alpha_i}(-1) \cdot v_{\lambda_0},$$

and these spaces agree, by (3.5). \square

Combining the short exact sequence (5.1) with two additional maps, which are isomorphisms (recall (3.33)), we obtain the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W(\Lambda_0) & & & & \\ & & \downarrow e_{\lambda_i} & \searrow & & & \\ 0 & \longrightarrow & W(\lambda_i) & \xrightarrow{e_{\omega_i}} & W(\Lambda_0) & \xrightarrow{\mathcal{Y}_c(e^{\lambda_i}, x)} & W(\lambda_i) \longrightarrow 0 \\ & & & & \searrow & \downarrow e_{-\lambda_i} & \\ & & & & & W(\Lambda_0) & \longrightarrow 0, \end{array}$$

and thus another formulation of Theorem 5.1, which uses only the principal subspace $W(\Lambda_0)$ (note that e_{α_i} is proportional to $e_{\omega_i}e_{\lambda_i}$, by (2.22)):

Theorem 5.2. For every $i = 1, \dots, l$ we have the following short exact sequence:

$$0 \longrightarrow W(\Lambda_0) \xrightarrow{e_{\alpha_i}} W(\Lambda_0) \xrightarrow{e_{-\lambda_i} \circ \mathcal{Y}_c(e^{\lambda_i}, x)} W(\Lambda_0) \longrightarrow 0.$$

As we recalled in Section 2, the vector space V_p and its subspaces $W(\Lambda_i)$ and $W(\lambda_i)$ are graded by weight and charge, and these gradings are compatible. We consider the multi-graded dimension of the principal-like subspace $W(\lambda_i)$, for each $i = 0, \dots, l$:

$$\chi_{W(\lambda_i)}(x_1, \dots, x_l; q) = \text{tr}_{W(\lambda_i)} x_1^{\lambda_1} \cdots x_l^{\lambda_l} q^{L(0)},$$

where x_1, \dots, x_l and q are commuting formal variables and $L(0)$ is the standard Virasoro algebra operator, introduced earlier. Note that

$$\chi_{W(\Lambda_0)}(x_1, \dots, x_l; q) \in \mathbb{C}[[x_1, \dots, x_l, q]].$$

As in [CLM2], in order to avoid the multiplicative factors $x_1^{(\lambda_1, \lambda_i)} \dots x_l^{(\lambda_l, \lambda_i)} q^{\frac{1}{2}(\lambda_i, \lambda_i)}$, we use the following slightly modified graded dimensions:

$$\chi'_{W(\lambda_i)}(x_1, \dots, x_l; q) = x_1^{-(\lambda_i, \lambda_1)} \dots x_l^{-(\lambda_i, \lambda_l)} q^{-\langle \lambda_i, \lambda_i \rangle / 2} \chi_{W(\lambda_i)}(x_1, \dots, x_l; q) \quad (5.3)$$

for $i = 1, \dots, l$, so that $\chi'_{W(\lambda_i)}(x_1, \dots, x_l; q) \in \mathbb{C}[[x_1, \dots, x_l, q]]$ (and $\chi'_{W(\lambda_0)} = \chi_{W(\lambda_0)}$).

Using the isomorphism (3.33)–(3.34) and (3.9), we easily obtain

$$\chi'_{W(\lambda_i)}(x_1, \dots, x_l; q) = \chi_{W(\lambda_0)}(x_1, \dots, x_i q, \dots, x_l; q) \quad (5.4)$$

for $i > 0$.

Following [Cal1] (see also [CLM1] and [CLM2]), from Theorem 5.1 (or equivalently from Theorem 5.2) we obtain a canonical system of q -difference equations for the graded dimension of the principal subspace $W(\lambda_0)$:

Theorem 5.3. *We have the following system of q -difference equations for $i = 1, \dots, l$:*

$$\chi_{W(\lambda_0)}(x_1, \dots, x_l; q) = \chi_{W(\lambda_0)}(x_1, \dots, x_i q, \dots, x_l; q) + (x_i q) \chi_{W(\lambda_0)}(x_1 q^{m_{i1}}, \dots, x_l q^{m_{il}}; q),$$

where $M = (m_{ij})_{1 \leq i, j \leq l}$ is the Cartan matrix of \mathfrak{g} .

Remark 5.1. The first term on the right-hand side comes from the map $\mathcal{Y}_c(e^{\lambda_i}, x)$ combined with (5.4); the factor $x_i q$ in the second term comes from the fact that $e_{\alpha_i} \cdot v_{\lambda_0} = e^{\alpha_i}$, which has λ_j -charge δ_{ij} and weight 1; and the expressions $x_i q^{m_{ij}}$ come from (3.6) with $\lambda = \alpha_i$, together with the analogue

$$e_{\alpha_i}(a \cdot v_{\lambda_0}) = \tau_{\alpha_i, c - \alpha_i}(a) \cdot e^{\alpha_i}, \quad a \in U(\bar{\mathfrak{n}})$$

of (3.34).

For any $n_1, \dots, n_l \geq 0$ we define $f_{n_1, \dots, n_l}(q) \in \mathbb{C}[[q]]$ by:

$$\chi_{W(\lambda_0)}(x_1, \dots, x_l; q) = \sum_{n_1, \dots, n_l \geq 0} f_{n_1, \dots, n_l}(q) x_1^{n_1} \dots x_l^{n_l}.$$

As in [Cal1], it is straightforward to show that the system of q -difference equations obtained in Theorem 5.3 has a unique solution in $\mathbb{C}[[x_1, \dots, x_l, q]]$ with the initial condition $f_{0, \dots, 0}(q) = 1$ ($f_{0, \dots, 0}(q)$ being the graded dimension of the subspace consisting of the elements of charge zero with respect to $\lambda_1, \dots, \lambda_l$), and to generate the solution. As usual, for any nonnegative integer n set

$$(q)_n = (1 - q) \dots (1 - q^n).$$

Following the same argument as in the proof of Corollary 4.1 in [Cal1], we obtain from Theorem 5.3, using the l equations in succession, the (multi-)graded dimension of the principal subspace $W(\lambda_0)$; then we invoke (5.4) to obtain the (multi-)graded dimensions of the principal-like subspaces $W(\lambda_i)$ for $i > 0$:

Corollary 5.1. *Let $M = (m_{ij})_{1 \leq i, j \leq l}$ be the Cartan matrix of \mathfrak{g} . We have*

$$\chi_{W(\lambda_0)}(x_1, \dots, x_l; q) = \sum_{\mathbf{n}=(n_1, \dots, n_l)} \frac{q^{\frac{\mathbf{n} M \mathbf{n}^T}{2}}}{(q)_{n_1} \dots (q)_{n_l}} x_1^{n_1} \dots x_l^{n_l} \quad (5.5)$$

(where each $n_j \geq 0$). Moreover, for $i = 1, \dots, l$,

$$\chi'_{W(\lambda_i)}(x_1, \dots, x_i; q) = \sum_{\mathbf{n}=(n_1, \dots, n_i)} \frac{q^{\frac{\mathbf{n}M\mathbf{n}^T + 2n_i}{2}}}{(q)_{n_1} \cdots (q)_{n_i}} x_1^{n_1} \cdots x_i^{n_i}. \quad (5.6)$$

Remark 5.2. Combining (5.6) with (5.3) gives us $\chi_{W(\lambda_i)}$ for $i = 1, \dots, l$.

As we mentioned in the Introduction, such formulas have been also studied in [DKMM], [KKMM1], [KKMM2], [KNS], [T], [FS1], [FS2], [G], [AKS], [Cal1] and [FFJMM].

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